Mesopotamian Calculation: Background and Contrast to Greek Mathematics

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Abstract

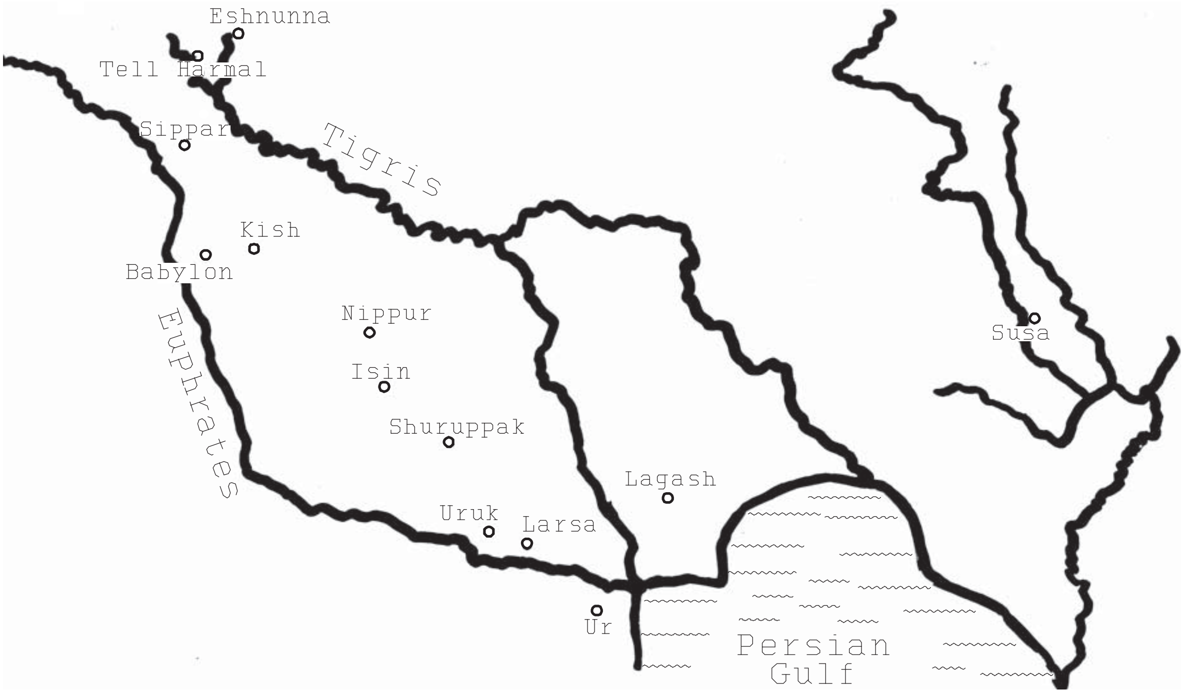
The fourth-millennium state formation process in Mesopotamia was intimately linked to accounting and to a writing system created exclusively as support for accounting. This triple link between the state, mathematics and the scribal craft lasted until the end of the third millennium, whereas the connection between learned scribehood and accounting mathematics lasted another four hundred years. Though practical mathematics was certainly not unknown in the Greco-Hellenistic-Roman world, a similar integration was never realized.

Social prestige usually goes together with utility for the power structure (not to be confounded with that mere utility for those in power which characterizes a working and tax/tribute-paying population), and until the 1600 BCE scribes appears to have enjoyed high social prestige.

From the moment writing and accounting was no longer one activity among others of the ruling elite (c. 2600 BCE) but the task of a separate profession, this profession started exploring the capacity of the two professional tools, writing and calculation. Within the field of mathematics, this resulted in the appearance of "supra-utilitarian mathematics": mathematics which to a superficial inspection appears to deal with practical situations but which, without having theoretical pretensions, goes beyond anything which could ever be encountered in real practice. After a setback in the late third millennium, supra-utilitarian mathematics reached a high point – in particular in the so-called "algebra" during the second half (1800–1600) of the "Old Babylonian" period.

Analysis of the character and scope of this "algebraic" discipline not only highlights the difference between theoretical and high-level supra-utilitarian mathematics, it also makes some features of Greek theoretical mathematics stand out more clearly.

Babylonian "algebra" was believed by Neugebauer (and by many after him on his authority) have inspired Greek so-called "geometric algebra". This story, though not wholly mistaken, is today in need of reformulation; this reformulation throws light on one of the processes that resulted in the creation of Greek theoretical mathematics.



Map of southern and central Mesopotamia, with ancient coast line and main rivers

Introductory observations

When we – the older members of the present congregation – first became acquainted with the history of early mathematics, it was conventional wisdom (derived from what Otto Neugebauer had proposed in the 1930s) that essential constituents of Greek geometry (in particular the so-called "geometrical algebra") had been created in answer to the "foundation crisis" caused by the discovery of incommensurability. Thereby, so it was thought, the numerico-algebraic knowledge of the Babylonians initially adopted by the Pythagoreans was saved. Then, around the mid-70s, this story was attacked by Hellenophile historians who knew much less than Neugebauer about Near-Eastern mathematics – and probably also less about the global undertaking of Greek mathematics, with its important links to astronomy.

None the less, 80 years of research have obviously changed our knowledge about the facts on which Neugebauer's theory was based. On one hand, the Greek "foundation crisis" has turned out to be a projection of that of the 1920s; on the other, Neugebauer himself already reformulated the idea about the transmission of Near Eastern knowledge in a paper from [1963], unnoticed by his critics. Moreover, present-day historiography of mathematics asks questions different from those prevailing the 1930s and the 1970s. When looking today for links and contrasts between Greek and Babylonian mathematics we therefore have to start anew.

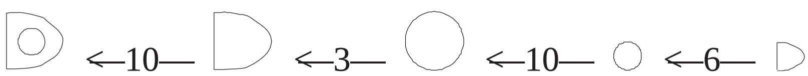
The first observation to make is that the very word “Babylonian” is somewhat inadequate. The city of Babylon only became important in the earlier second millennium BCE, and only from then on does it make sense to characterize the southern and central part of Mesopotamia as “Babylonia” – still excluding the Assyrian north. Since much more is now known about the mathematics of the third and fourth millennium than forty years ago, it is therefore more fitting to speak of “Mesopotamian mathematics”. As we shall see, it might also be more fitting to specify and speak of “calculation” instead of “mathematics” in broad generality; none the less, let us accept the less specific but conventional name “Mesopotamian mathematics”.

The social support

Mesopotamian mathematics emerged together with a pictographic script (later transformed into the mixed logographic-syllabic “cuneiform” script) at the beginning of the “proto-literate” period (c. 3200 BCE[1]). Together, the two techniques were essential in the process of state formation (around the city state Uruk), where they appear to have allowed the transformation of an earlier redistribution system into a legitimizing ideology of “social justice” based on numerical “just measure”, both in the distribution of land to high officials (in arithmetical proportion in agreement with rank) and in the allocation of food to workers.

The main constituents of the mathematics of the period are (1) a system of metrologies, (2) an accounting system, and (3) basic area measurement. The most important metrologies[2] are:

(a) The grain system:



(the smallest unit is to the right, the numbers indicate the step factors);

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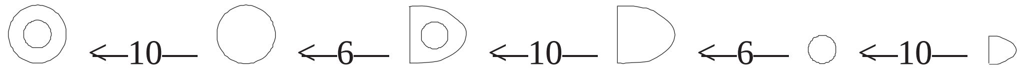
[1] I follow the “middle chronology”, as used in [Liverani 1988]. In order to facilitate comparison with work using one of the other possible chronologies (“long”, “short” etc.) I also indicate period or ruler names.

[2] The fundamental reference for proto-literate metrologies is [Damerow & Englund 1987], on which the following description is built.

(b) the counting system:

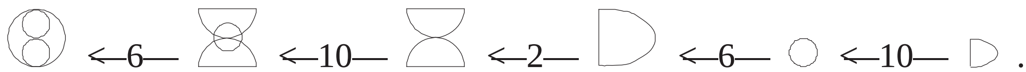


(stands for 1; as we see, the number system is already sexagesimal – or rather, as also later on, seximal-decimal, as the Roman number system is dual-quintal);

(c) the area system:

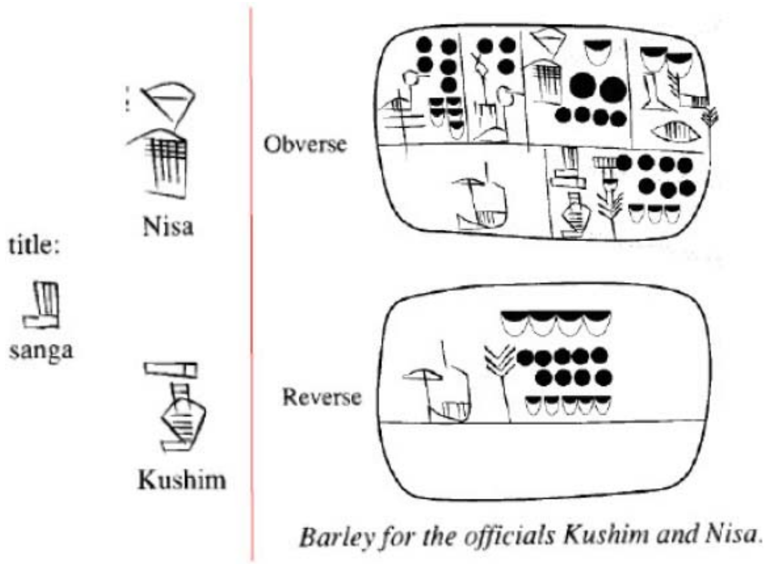
(this was geared to the length system – lengths themselves were written in the counting system; standing for the square on , that is, on 10 times the length unit).

Besides, a particular “bisexagesimal” counting system was used for the counting of bread or grain rations, perhaps also for portions of dairy products:



Similarly, particular systems were in use, for instance for malted barley, derived from the normal grain system; and finally a calendar used for administrative (not cultic) purposes, in which each twelve months counted as a year, and each month was counted as 30 days irrespective of whether its real length was 29 or 30 days.

Accounting was really a *system:* it employed a fixed format, in which the obverse of tablets contained for instance allowances to single individuals or the various ingredients going into the production of a batch of beer, and the reverse the total.[3]



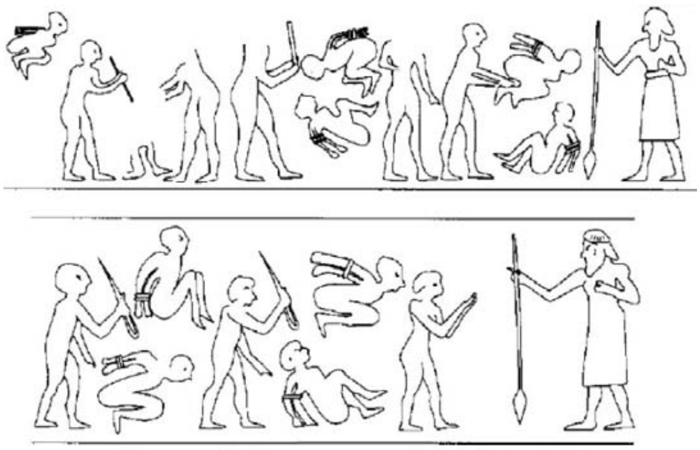
From [Nissen et al 1993: 39]

The gearing of the area to the length metrology already shows that rectangular areas were found as

[3] Numerous examples can be found in [Nissen, Damerow & Englund 1993]; cf. also [Green 1981].

the product of length and width (with the difficulties that this entails because of the incongruity of higher units, which must have represented mathematically normalized versions of older “natural” measures). A school text (see imminently) betrays that the area of approximately rectangular quadrangles were determined by means of the “surveyors’ formula”, as average length times average width.

Some 85% of the texts from the period are administrative texts. The remainder consists of “lexical lists” used to train the script. A few texts apparently belonging to the former group are not genuinely administrative but “model documents”: school texts that are distinguishable from real administrative documents by the absence of an official’s seal and by the appearance of rounder numbers than can be expected in real-life management. Literary texts are absent, as is every kind of mathematics not directly linked to administration.[4] Even boasting on the part of the rulers (a priesthood, not a king or a warrior aristocracy) is not expressed in writing but only in the pictures on seals (see two typical examples in the figure) and in prestigious temple buildings. Writing and computation was wielded by the priesthood, but only as a tool, not used for social identity construction or to express professional pride.



From [Nissen et al 1993: 16]

Temple building must have involved a fair measure of practical geometrical knowledge, but evidence from later times suggests that this knowledge was the possession of master builders and did not communicate with the mathematics of the literate managers; in any case, we have no evidence for its character except what is offered by temple groundplans (that is, nothing specific). What we know about is only the mathematics of the literate tradition, which was purely

[4] It may need emphasis that mathematical astronomical even in the most rudimentary sense was millennia into the future. Karl Wittfogel’s “cloak of magic and astrology [...] hedged with profound secrecy [...] bulwarking the superior power of the hydraulic leaders” [1957: 30] is pure fantasy in as far as applied to Mesopotamia (as is much of the rest of what he says in this undeservedly famous book).

computational, and at the same time purely utilitarian.[5]

As an aside, we may compare with the situation of proto-classical Greece. Even if we are generous and accept the stories of Thales as the founder of Greek geometry, his foundation of is of course far removed in time from the beginning of the *polis* system, though contemporary with the Solon reforms in Athens. The genesis of the *polis* did not in any way depend on mathematics. One aspect of the Solon reform, however, has a slight affinity with mathematics – his reform of metrology. In the 590s BCE, Greece was already fully immersed in monetary economy – indeed, the debt crisis this monetary economy had caused was one of the reasons that the reforms were necessary. None the less, the reform as described by Aristotle in *The Constitution of Athens* 10 (trans. E. G. Kenyon *in* [Barnes 1984: II]) does not seem to build on mathematical thought at the abstraction level of the proto-literate manager-priests:

> [...] He also made weights corresponding with the coinage, sixty-three mines going to the talent; and the odd three mines were distributed among the staters and the other values.

The Mesopotamian proto-literate state was displaced in the initial third millennium BCE by a system of city states competing not least for water resources, which gave their war leaders the opportunity to take on a royal role. The first couple of centuries has left us no written documents, but around 2700 BCE we find the earliest royal inscription, and an abundant century later in the city state Shuruppak the first evidence for the existence of a separate profession of *scribes*, working both in the state bureaucracy (without being masters of the state, as the proto-literate manager-priests had been) and in the elaboration of private economic contracts.

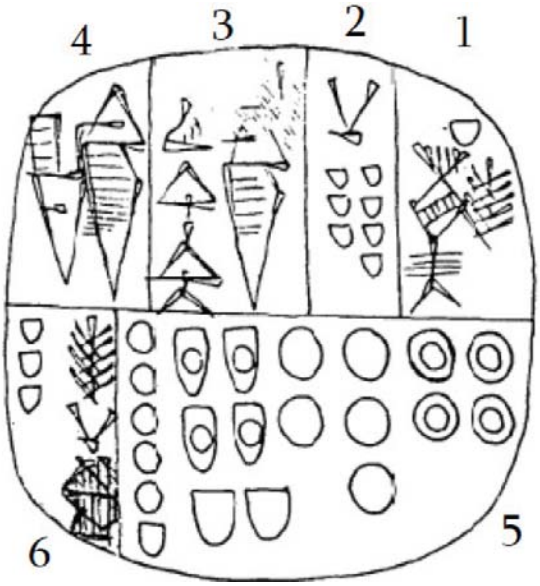
We possess a large number of school texts from Shuruppak – some of them, not least large lexical lists going back to the proto-literate tradition, apparently *de luxe* versions made for scribes well into their professional career “in memory of good old school days” (Aage Westenholz, personal communication). This, if really true, would be one attestation of professional pride. Another confirmation (also indirect) is the appearance of two new genres; *literature* (a hymn and a

[5] From the whole duration of Mesopotamian culture we also find decoration making use of interesting patterns: regular pentagons and hexagons, complicated knots drawn with a single continuous line – cf. [Friberg 2007: 416–418]; but apart from the submission of the regular polygons to *computation* in the Old Babylonian period, we have no hint that this interest communicated with the literate mathematical tradition, or that it was submitted to scrutiny for mathematical principles.

proverb collection) and *supra-utilitarian mathematics* – that is, mathematics that looks as if it has to do with the utilitarian tasks of scribes but at closer inspection turns out to go far beyond what could ever present itself in professional practice. Both genres can be understood as explorations of the carrying capacity of the two professional tools, writing and computation.

One example of supra-utilitarian mathematics runs as follows [Høyrup 1982]:

1. Grain, 1 granary,
2. 7 sila
3. each man receives.
4. Its men?
5. 45,42,51
6. 3 sila of grain left on the hand.



The sila is a measure of capacity, c. 1 litre; the “granary” is a standard expectation – perhaps a genuine unit – of 40·60 gur (“tuns”), each gur being 8·60 sila. We are thus to distribute 1152000 sila in portions of 7 sila; the result of the division is 164,571 men, with a remainder of 3 sila left on the counting board (called “the hand”).

The problem is so far removed from anything that could be encountered in practical life that the Raymond Jestin [1937: 24] in his edition of the text (understanding only that “men” were dealt with, in number 164,571), wondered what could be meant – there were fewer inhabitants in the city. The merit of the problem is that 7 does not divide any of the metrological factors, and that the immensity of the number makes the calculation difficult as well as striking.

The problem type appears to have been fashionable; another tablet from Shuruppak contains the same problem (but with an erroneous or incomplete solution and thereby reveals the procedure that is used), while a text from Ebla in Syria, of somewhat later date, contains a division of 260,000 by 33 [Friberg 1986: 16–22]. Interestingly, the methods used to solve the Ebla problem and the Shuruppak problems are different. As Jöran Friberg [1986: 22] formulates the matter

> the “current fashion” among mathematicians about four and a half millennia ago was to study non-trivial division problems involving large (decimal or sexagesimal) numbers and “non-regular” divisors such as 7 and 33.

Even though one may perhaps question Friberg's introduction of a social category "mathematicians", exploration – to some extent systematic exploration – of the potentials of computation beyond the realm of the immediately useful is certainly involved.

In the mid-24th century, southern Mesopotamia was united in a single territorial state, first for some decades by the ruler of Uruk, then until c. 2200 BCE under the Akkadian king Sargon and his dynasty.[6] For a while, Sargon's successors also subdued large parts of the Syrian area (etc.), for which reason one often speaks of a Sargonic "empire". Literature (supra-utilitarian writing) was understood by Sargon to be a useful vehicle for propaganda – his daughter Enḫeduanna rewrote ancient myths so as to fit the new political situation and is thus the earliest poet in history to be known by name. As may be easily guessed, supra-utilitarian mathematics could have no similar function, but utilitarian mathematics was reshaped: not only was accounting made on a larger scale than before [Foster 1982], a "royal" metrology was superimposed on the partially divergent local metrologies. Within the scribe school, however, we have evidence that supra-utilitarian mathematics lived on. To find the area of a rectangular field from its length and width was certainly of real use; the inverse problem, to find one side from the area and the other, was not – yet all the same it was well trained (see the overview of Sargonic mathematical school texts in [Foster & Robson 2004]).[7]

The Sargonic empire lasted no longer than the British world empire (if we count that generously, from Nelson's victories to the decolonization of India and Africa). From c. 2200 BCE, the political landscape was again dominated by city states (and by incursions from the East). The best known ruler from the following century is Gudea of Lagash, in particular because of his building activities and the inscriptions in statues where he boasts of these, many of them on statues (see [Edzard 1997]). One of these shows him in the role of architect, with a writing tablet with stylus showing the ground plan of the temple on his lap, and a measuring rule. It is thus evidence of the kind of practical geometry which

[6] Akkadian is a Semitic language, whose main dialects in later times are Babylonian (spoken in the south) and Assyrian (spoken in the north). Its form during the Sargonic dynasty is referred to as "Old Akkadian". Already in Shuruppak, personal names shows it to have been present in the otherwise Sumerian-speaking area.

[7] Because of the character of the metrologies, neither problem was as straightforward as we may be tempted to believe – we get an impression if we think of the area as expressed in acres and of the linear dimensions as expressed in fathoms, feet and inches.

went into prestige building, but even less informative about its substance than the building remains themselves – the text on the statue, not informative at all on this account, is published and translated in [Edzard 1997: 31–38].

Much more can be said about what happened in the territorial state ruled during the 21st century by the "Third Dynasty of Ur" (known as "Ur III") – between 2075 and 2025 BCE ruling also over central Iraq and Susa in the Zagros region, thereby becoming an "empire". The beginnings, the time of the founder Ur-nammu and the first two decades of his son Shulgi, offer nothing spectacular to our topic. However, in the immediate wake of a military reform connected to the establishment of the empire (as cause or as consequence?), an administrative reform gave rise to two mathematical innovations, one of which is still with us today.

The crux of the reform was a reorganization of labour. At least in the Sumerian core of the empire, most workers were organized in labour troops guided by overseer-scribes. These were responsible for the produce of their crew calculated meticulously and accounted for in agreement with fixed (harsh) norms.

One innovation concerned *the accounting*. It can be compared (in spirit and efficiency, not in actual shaping) to the Renaissance introduction of double instead of single entry book-keeping. It kept track of debits (for workers allocated, workers borrowed from other overseers, etc.) and credits (produce, workers lent to other crews or booked out because of illness, death or flight, etc.).[8] The system survived Ur III and was used (within a different economic framework) during the Old Babylonian period (see presently), and then disappeared

[8] It also kept track of the deficit of the overseer as it grew over the years (it almost invariably did), which, if it could not be covered from his possessions at his death, would result in the bereaved being taken into state slavery – see a document in [Englund 1991: 268]. The article deals with Ur III labour management in general; a more extensive presentation is given in [Englund 1990].

The calculation was behind the other innovation, the one that still survives: the sexagesimal place-value system – a floating-point place-value system with base 60. I borrow from [Høyrup 2002a: 17*f*]:

> Its function derives from the fact that the metrological sequences were not always arranged sexagesimally: thus [...] the nindan is subdivided into 12 kùš and the kùš into 30 šu.ši – no factor 60 occurs. If, for instance, a platform had to be built to a certain height and covered by bricks and bitumen, a "metrological table" had to be used to transform the different units of length into sexagesimal multiples of the nindan and kùš, allowing the determination of the surface and the volume in the basic units sar and [volume] sar. A list of "constant coefficients" (igi.gub) would give the amount of earth carried by a worker in a day over a particular distance, the number of bricks to an area or volume unit, and the volume of bitumen needed per area unit – all expressed in basic units (if no transformation into basic units had taken place, different coefficients for the bitumen would have had to be used for small platforms whose dimensions were measured in kùš and for large ones measured in nindan). With these values at hand the number of bricks and the amount of bitumen as well as the number of man-days required for the construction could be found by means of sexagesimal multiplications and divisions – once again facilitated by recourse to tables, this time tables of multiplication and of reciprocal values. Finally, renewed use of metrological tables would allow the calculator to translate the results of the calculations into the units used in technical practice.[9]

We do not know whether the place-value principle was *invented* during Ur III – as shown by Marvin Powell [1976], the idea appears to have been "in the air" for centuries (it was probably an easy transfer from the counting board, in use since Shuruppak at least); however, all texts that suggest the idea contain errors, suggesting that the *system* was not there. And indeed, the system was useless unless supported by mass production of arithmetical and technical tables[10]

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[9] The nindan is the basic unit for horizontal distance (c. 6 m), the kùš (cubit, c. 50 cm) that for vertical distance. The sar is 1 nindan^2 when used as a surface unit and (1 nindan^2·kùš) when measuring volumes.

The metrological and arithmetical tables were learned by heart in school; later on, scribes had no need to consult the physical specimens.

[10] The fundamental arithmetical tables listed reciprocals and products (technical constants were chosen so as to have a finite and mostly simple sexagesimal reciprocal, and so division could be performed as multiplications by the reciprocal). At least in Old Babylonian times, lists of squares and of the square and cube roots of perfect squares respectively cubes were also found (and even a list called "equalside one appended", confronting n and $n^2 \cdot (n+1)$. *They* were evidently not needed for the usefulness of the place-value notation, but the existence of square tables (in length and area metrology) from Shuruppak and the Old Akkadian period suggests their place-value equivalent to have

and mass teaching of how to apply them – which on the other hand made no sense if the system was not in use. Only a centralized decision enforced by a centralized power could get around this egg-and-hen problem – and this is likely to have happened as part of Shulgi's administrative reform.[11]

In hymns written in his name king Shulgi boasts (among various incredible feats) of having gone to school from early age, where he learned writing; only three Mesopotamian kings do as much, and only for the last of them (Assurbanipal, on whom below) do we have evidence that the claim is justified. Shulgi also learned mathematics – that is, subtraction, addition, counting and accounting.[12]

Not a word about multiplication or reciprocals! And certainly not about any more complex matters, except to the extent they are supposed to inhere silently in accounting (as multiplication may perhaps have been). Striking as this is, it is probably symptomatic. Not only are the only school texts known from the time once again model documents, as had been the case before the emergence of the scribal profession; this could after all be the result of excavation accidents. More telling is what can be derived from the mathematical terminology of the subsequent Old Babylonian times, when the language of the school had become Akkadian, whereas that of Ur III was still Sumerian: terms that concern *calculation* may appear in syllabic Akkadian but also often be written with a "Sumerogram", that is, a word sign originally belonging within the Sumerian script but in Old Babylonian times probably read in Akkadian (as the originally Latin word sign *viz* is now read "namely" by English speakers). But in order to *structure a problem* – with a statement presenting givens and question, and a procedure description with a well-defined architecture and a way to announce results – a whole supplementary terminology is required. The constituents of this terminology are invariably written in syllabic Akkadian within the mathematical

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been created together with the implementation of the place-value system.

11 The system was used for intermediate calculations, which have normally disappeared, and in the tables that were used for training it, which are very difficult to date paleographically. However, a few tables have been found together with dated Ur III texts, for which reason use during Ur III is now certain.

During the Old Babylonian period, place-value numbers were also used in mathematical school texts, and in the first millennium BCE in astronomical tables. Because of its floating-point character, the place-value system could evidently not serve in final accounting or other legally meaningful documents.

12 Hymn B, l. 13–19, ed. [Castellino 1992: 32]. Castellino's translation and commentary miss the mathematical points completely.

texts,[13] even when the same Akkadian words might be written logographically in other contexts.[14] It appears that the Ur III school, concentrating solely on producing obedient and efficient overseer scribes, had simply eliminated problems (utilitarian as well as supra-utilitarian) from the curriculum, avoiding thus even that modicum of independent thought that is needed to *find* a method instead of following a prescribed procedure.[15]

In 2025 the periphery revolted successfully, and after another 20 years the Ur III state collapsed, provincial rulers making themselves independent (the top-heavy bureaucracy being probably one of several factors in the demise). The following four centuries are spoken of as the "Old Babylonian period". Already during its early phase, the role of "the palace" in the economy became less dominant. Well before 1800 BCE, a private sector had developed – even palace land, still a major factor in the economy, was farmed ut, not cultivated by labour troops [Leemans 1975]. A kind of banking had also developed along with a system allowing trade in land, otherwise not allowed to leave the kinship group [Stone 1982].

From our present perspective it is important that the scribal profession had regained autonomy (probably gained more than it ever had). Firstly, scribes might be employed by large-scale private economic actors, or even write private letters as free-lance street-corner scribes – a category that had been as non-existent before as the private letter itself. For mathematics, however, ideological autonomy was what mattered.

An overall ideological characteristic of the period[16] was the emphasis on individuality or personality. In the case of the scribes, this gave rise to an enhanced professional self-awareness – if not in reality, for which we have little direct evidence, then at least (and that is still what is important for our purpose) in the ideas about true scribehood which the scribe school tried to inculcate.

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[13] Toward the very end of the Old Babylonian period, a few exceptions turn up – but only toward the end, when new Sumerograms or pseudo-Sumerograms had been invented,

[14] The argument is complex, and cannot be developed here. See [Høyrup 2002b] or (more briefly, but drawing heavily on preceding chapters) [Høyrup 2002a: 376–378].

[15] At least in private, Robert Englund would speak of a "Kapo economy" in order to summarize the findings reported above (note 8). Even the Kapo of the KZ camp was of course not suppose to think independently (and actually, according to the one former Kapo I have known, thinking only independently of how to survive).

[16] Obviously concerning only the literate stratum and those on whose behalf writing was made or expressive art was created – we have no other sources for ideology, neither direct nor indirect.

These are reflected in a number of so-called “edubba texts” which advanced students had to study, copy and understand.[17] They may contain dialogues between the master and a student or between students. What is needed for true scribehood or “humanity” (*sic*, that is the meaning of the Sumerian term for the quality) is supra-utilitarian knowledge. Reading and writing Akkadian was trivial, the scribe had to read, write and *speak* Sumerian, which nobody but other scribes would understand (the Latin of recent Western European centuries comes to mind). Knowing the syllabic values of cuneiform signs was insufficient even though these were fully suitable for expressing the spoken language – the requirement was familiarity with all the logographic values, including some so secret that we do not understand what is spoken about. And also needed was familiarity with music and mathematics – unfortunately not too specific. One text [ed., trans. Sjöberg 1975: 167] speaks of multiplication, reciprocals, technical coefficients, accounting and balancing of accounts, division of property and delimitation of fields; others [Friberg 2000: 153] also refer to mensuration and construction of trapezia (or trapezoids).

From excavations we know about the normal mathematical curriculum (in Nippur, but the relative uniformity of scribal habits from the whole Babylonian area allows us to generalize).[18] At first students had to copy and recopy (and thus learn by heart) the “metrological lists”, lists of metrological units and their multiples. Later “metrological tables” were trained, were each entry from the metrological lists was confronted with its sexagesimal equivalent in the tacitly assumed basic unit (cf. above, p. 9). Then followed the fundamental arithmetical tables, and then – now we are at the level where Sumerian literature was taught! – arithmetical squares and the determination of simple areas.

The edubba texts, we remember, speak of calculation, technical constants and surveying, and about the partition of fields and thus geometrical calculation; but that is where our information ends about which kind of mathematics could be considered “humanist”. On the other hand, a large number of texts contain sophisticated mathematics. Unfortunately, most of these texts have been excavated illegally and have been bought by museums or private individuals on the antiquity (or in recent decades, where UNESCO rules have been introduced, black) market. Their appurtenance to the Old Babylonian period can only be derived from orthography and script, and their precise origin often cannot be

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[17] “Edubba” means “tablet house”, that is, “school”. Various specimens are published with translation in [Kramer 1949] and [Sjöberg 1972; 1973; 1975], cf. also [Sjöberg 1976].
[18] See [Robson 2002] and [Proust 2008].

determined with certainty. Some texts *have* been excavated regularly, but none have been found in an unambiguous school contexts. All we can say is that the *format* of the texts is that of school problems – often with an introductory statement in the first person singular, a formula opening the procedure description, which is stated in the imperative or the second person singular and perhaps with references to the statement as something "he" has said (the actors thus being the master, the student and the instructor, known from the edubba texts). From the Nippur excavations it appears, however, that the more sophisticated of these problems went beyond what was taught to normal students even at the advanced level. In so far it is still distressingly true what Neugebauer wrote in [1934: 204], namely that

> we still know practically nothing about how Babylonian mathematics was situated within the overall cultural framework.

The sophisticated problems may have been taught to a subgroup of students; to such students as prepared for teaching mathematics[19]; or they may just have been made in school format without any intention that they should really be taught. We have no clues.

Format, indeed, need be no more than evidence for *the soil* from which an intellectual activity has grown. We may think of Oresme's discussion of the possibility that the earth and not the whole universe around it rotates every 24 hours. It belongs within a treatise written for the king. None the less, its whole style is that of a university *quaestio*, and thus ultimately of a university disputation.[20] The point is also illustrated by the adjacent "formal door", found in the park of the Beijing Tsinghua University, once belonging the emperor. It *is* not part of a wall, but the artist would never have imagined it if walls with doors had not existed. In this sense, the sophisticated problems may be seen to have grown out of the school environment, irrespective of whether, and to whom,

[19] This would be similar to one of the likely purposes of Italian abbacus algebra, also no part of the ordinary abbacus school curriculum.

[20] *Le Livre du ciel et du monde*, Book II, ed. [Menut & Denomy (eds) 1968: 518-538]. The English translation is to be avoided, it misses many of the hints of university style.

they were intended to be taught.

In the next section we shall return to the mathematics of these sophisticated problems, which indeed represent what is normally known in histories of mathematics as "Babylonian mathematics". For now it suffices to take note of what *can* after all be said about their geographical origin and chronology.

One group of texts (discussed in [Friberg 2000]) comes from Ur. The date is uncertain, but almost certainly Old Babylonian, and probably relatively early – the nineteenth century BCE is plausible. In any case, a somewhat rudimentary problem format can already be recognized, but the problem types that dominate the 18th-17th-century texts are absent.

A small lot of mathematical tablets have been found in the royal archives of Mari in the north-west [Soubeyran 1984]. They date from the decades preceding the conquest of the Mari state by Hammurapi of Babylon in 1758 BCE, and there is no particular reason to believe they represent the whole gamut of local mathematical culture. Most are tables of multiples and reciprocals, and thus within the a tradition imported from Ur III.[21] One, however, is of a different character. It is not formulated explicitly as a problem, but at least it shows that the calculator-scribe could use his mathematics for something which looks like fun. The text contains the earliest known version of the "chess-board problem", continued doublings starting with a single barley grain and passing to metrology for larger quantities (but as in all versions prior to the diffusion of the chess game, with 30 doublings).[22]

Much more informative are the texts found in Eshnunna in the north-east, written between c. 1790 and c. 1775 BCE (Eshnunna was conquered by Hammurapi in 1761 BCE).[23] They contain not only genuine problems but also deliberate work on creating a *format* for formulating problems (cf. [Høyrup 2002a: 319–326]); this format is rudimentary in the earliest text but more richly developed in the later ones. Some of the problems concern what seems to be real-life problems involving technical coefficients, others are clearly supra-

[21] Mari had not been fully under Ur III rule, but the Ur III techniques obviously radiated further than political control.

[22] See in particular [Boyaval 1971] (a papyrus from Roman Egypt), [Folkerts 1978: 51*f*] (in *Propositiones ad acuendos iuvenes*, a Carolingian problem collection), and [Saidan 1978: 337], the statement that "many people ask [...] about doubling one 30 times, and others ask about doubling it 64 times" (al-Uqlīdisī, Damascus, CE 952/53).

[23] A text from c. 1790 is in [Baqir 1950a], others from c. 1775 are in [Baqir 1950b; 1951; 1962] and [al-Rawi & Roaf 1984].

utilitarian, inaugurating in particular the genre that has come to be known as "Babylonian algebra" (on which below).

Many of the problems, however, open in a way that differs from what we find in other, probably later supra-utilitarian texts: not by an anonymous speaker (according to the format, the teacher) stating that "I have done so and so" but in riddle style, "If somebody asks you thus: 'I have done so and so ...'". One problem (IM 53957, [Baqir 1951: 37], corrections [von Soden 1952: 52]) is a mock calculation, in which the result is presupposed in the procedure – typical of riddles that have not yet been fully absorbed into a properly mathematical practice; it starts as follows:

> If somebody asks you thus: to two-thirds of my two-thirds I have added hundred s i l a of barley and my two-thirds, 1 g u r was completed. The *tallum* vessel of my grain corresponding to what?

This is strikingly close to problem 37 of the Rhind mathematical Papyrus [ed., trans. Chace et al 1929: Plate 59] – closer than could happen by chance:

> Go down I [a jug of unknown capacity] times 3 into the *ḥeḳat*-measure, $^1\!/_3$ of me is added to me, $^1\!/_3$ of $^1\!/_3$ of me is added to me, $^1\!/_9$ of me is added to me; return I, filled am I [actually the *ḥeḳat*-measure, not the jug]. Then what says it?

The Egyptian problem appears to be the only one in the complete corpus of Pharaonic texts (not only mathematical texts) which makes use of an "ascending continued fraction", typical of Semitic languages (Arabic as well as Akkadian). It seems reasonable to conclude that we are here confronted with a riddle travelling (probably with Semitic-speaking merchants) between the two high cultures, adopted in both places into the regular mathematics of the scribe schools.[24]

This, together with the pervasive formula "If somebody" and the repeated doublings from Mari, suggests that the scribe school, in search for supra-utilitarian problems that might serve to demonstrate scribal "humanism", borrowed from oral riddle traditions. The filling and doubling problems may both come from a merchants' environment. The "algebra", however, being based on measurable line segments and areas, must have been borrowed from lay (and, according to other evidence, Akkadian-speaking) surveyors.

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[24] In Egypt, the solution has also been regularized, making full use of the unit fraction technique; later Babylonian versions of the problem are normalized to such an extent that the family resemblance to the Egyptian problem would not be recognizable without the Eshnunna precedent

Comparison with sources from classical Antiquity and the Islamic Middle Ages suggests that a restricted number of riddles about squares and rectangles circulated within a Near Eastern surveyors' environment: about a single square, where the sum of or difference between the area and the side or "all four" sides is given; about two squares, where the sum of or difference between the areas is given together with the sum of or difference between the sides; about a rectangle, for which the area is given together with one of the sides, the sum of the sides or their difference; and a few more.[25] Most of them are mixed second-degree problems, and can be solved only by means of a (geometric) quadratic completion. The only exceptions are the two problems about a rectangle where the area and one side are known, present already in the Sargonic curriculum. Since all the others are absent from the Sargonic record, it seems plausible that the trick – labelled "the Akkadian [method]" in one late Old Babylonian text (below, p. 24) – was invented between 2200 and 1800 BCE.

Within the context of the scribe school or growing out of it, this restricted list of standard problems was the basis for the development of an advanced discipline. It seems (but because of the difficult dating of mathematical texts from the southern region we cannot be sure) that it started to flourish in the former Sumerian heartland (Larsa and Uruk are likely find-spots for the texts in question) after Hammurapi's conquest of Eshnunna. Later on, probably in the 17th century BCE, we also see it flourish in the north – a number of important texts appear to come from Sippar. From late Old Babylonian Susa comes another important text group, to which belong very sophisticated problems as well as explicit didactical expositions probably reflecting what was only explained orally elsewhere (translated below).

Already some twenty years after Hammurapi's conquest of Eshnunna and Mari, serious rebellions started in the north as well as the south, and from around 1720 the extreme south was independent; it seems that scribal high culture did not survive in the area.

Around 1600, the Hittites raided Babylon, which was the end of the Hammurapi dynasty and of the Old Babylonian socio-cultural complex. The Kassites, already present in the area, probably both as labourers and as soldiers, took over power but neither statal legitimization nor scribal culture. The ratio between city and countryside dwellers fell to the level preceding state formation. The scribe school with its integration of literate and calculational curriculum

[25] The arguments for this are complex, and there is no space for them here. But see [Høyrup 2001].

disappeared. *Learned* scribes – those trained in Sumerian language and literature – were from now on taught within scribal families (*real* families). Some surveying and administrative activity went on, but as far as can be seen from the sources from the time when written documents in larger quantity reappear, numeracy (including the use of the place-value system) is likely to have been the responsibility of less learned though literate staff. In any case, the advanced mathematics of the 18th and 17th centuries did not survive, and could not be resuscitated when the late Assyrian rulers allied themselves with the learned scribes of their time, using them as counsellors (not least in matters astrological) and as producers of imperial ideology. In the mid-7th century BCE Assurbanipal, the third Mesopotamian ruler who boasted of literacy (and in his case we know it to have been not wholly untrue), declares [Ungnad 1917: 41*f*] that he is able to "find reciprocals and make difficult multiplications" – a quite modest claim, we may find, when advanced by somebody who also asserts to be able to read tablets from "before the flood", that is, from pre-Sargonic times.

Something superficially similar to the Old Babylonian discipline finally turns up in Late Babylonian times (probably in the fifth century BCE). We have one text containing rectangle problems where the area and the sum of respectively the difference between the sides is known, and problems about two squares with given area difference [ed. Friberg 1997]. As we observe, these all belong to the old collection of riddles;[26] moreover, things are now stated in area metrology, showing that the carrying tradition must have been real surveyors, even though the owner of the tablet was a scholar-scribe.[27] Finally, there is discontinuity in the use of Sumerographic equivalents of Akkadian words, suggesting that a recent re-Sumerianization had been undertaken by the scholar-scribes when they took up mathematics once again. We can only guess at the reasons that they did so after a millennium; somehow, it seems, the environment had become aware (perhaps because of its being centrally involved in development of mathematical astronomy) that mathematics belonged to the scribal tradition.

However that may be, we have too few mathematical texts from the epoch to pursue questions about the mathematical practice and its social substratum,

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[26] There is ample evidence for the survival of the surveyors' riddle tradition into Greek Antiquity (to which we shall return), the Islamic Middle Ages with their offset in the Latin and Italian medieval world, and even in Sanskrit Jaina mathematics. Discussing this would lead too far, but see [Høyrup 2001].

[27] Probably the same Šamas-addina who identifies himself as the owner of related text ([Friberg, Hunger & al-Rawi 1990: 545], cf. [Robson 2008: 227–237]). Neither he nor his scribal family would have the least to do with practical surveying.

and also too few to say anything about the cognitive character of this practice itself. This also holds for the last epoch from where we have mathematical texts – the Seleucid period (third and second century BCE), for which reason I shall say no more about Late Babylonian mathematics.

Old Babylonian "algebra"

Let us turn instead to Old Babylonian "algebra", that discipline which was taken by Neugebauer to have inspired Greek "geometric algebra" (my quotes, not his).

When Ernst Weidner and a few other Assyriologists started the decipherment of the Babylonian mathematical terminology from 1916 onward, the numbers provided the key. If an operation on 5 and 6 produces 11, it was supposed to be an addition; if the outcome was 30, the operation had to be a multiplication. As the task was taken up on an immensely larger scale by Neugebauer and François Thureau-Dangin from the late 1920s onward, the same principle was followed. That was not only natural but a necessity, and it produced important insights. It was certainly recognized that the texts appeared to deal with the sides and areas of rectangles and squares – but this was supposed to be metaphorical talk (even our "square" numbers do not possess four sides).

Unfortunately, the initial successes of the method barred understanding of its shortcomings; in particular it was not sees that an operation that could be characterized as *an* addition was not thereby simply arithmetical addition (etc.). Analysis of the total corpus of advanced mathematical texts shows that they make use of two different additive operations; two subtractive operations; four multiplicative operations; and that they distinguish two different halves.[28]

These distinctions cannot be explained within the purely arithmetical interpretation that resulted from the first identification of the operations. Instead, the squares and rectangles and their inherent geometry have to be taken seriously. Let us first look at the simplest of all mixed second-degree problems, BM 13901#1:[29]

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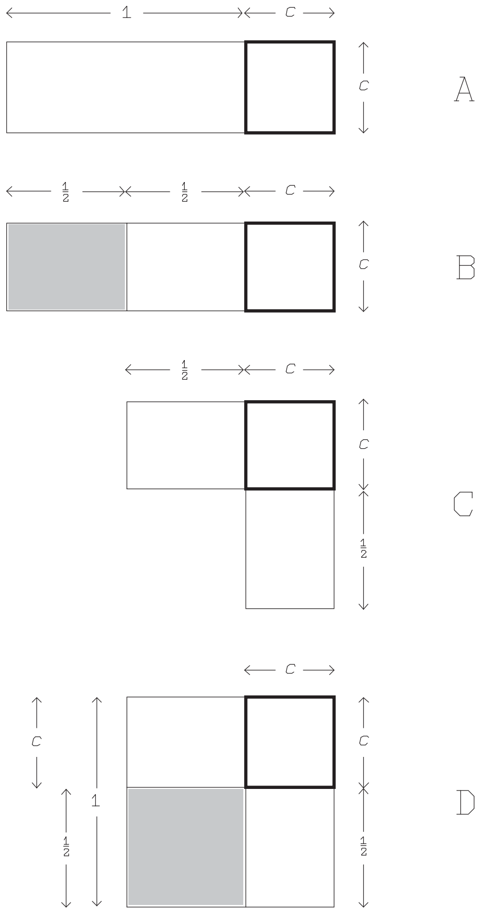
[28] These are *not* synonyms, but *operations* that are kept strictly apart. Many of the operations can then be indicated by two or several synonymous terms (beyond the possibility to write them syllabically or with logogrms).

[29] The text was first published by Thureau-Dangin [1936], then (translation and transliteration only) in [Neugebauer 1935: III]. Here following [Høyrup 2002a: 50–52].

As can be seen from the many square brackets, the tablet is damaged; however, the language is so standardized and so repetitive that the reconstructions are not subject to doubt.

1. The surfa[ce] and my confrontation I have accu[mulated]: 45´ is it. 1, the projection,
2. you posit. The moiety of 1 you break, [3]0´ and 30´ you make hold.
3. 15´ to 45´ you append: [close by] 1, 1 is equal. 30´ which you have made hold
4. in the inside of 1 you tear out: 30´ the confrontation.



The procedure of BM 13901 #1, in slightly distorted proportions.

The problem deals with a square, of which the Babylonians thought primarily as a square frame constituted by the confrontations of equal sides, while we think of its as a surface contained by a boundary. Our square thus *is* its area and *has* a side, whereas that of the Babylonians *had* an area and *was* [parametrized by and hence identified with] its side, spoken of as the "confrontation". The (measuring numbers of the) area and the side of this square are thus added with a symmetric operation that allows the addition of measuring numbers (and hence of magnitudes of different dimension), with result 45´ (= $^3/_4$). In order to make this concretely meaningful and allow geometric manipulation of the data, the confrontation is provided with a "projection 1", a width 1 which transforms it into a rectangle with the same measuring number – see the adjacent figure.[30]

I present this as well as following texts in "conformal" translation – that is, different terms are always translated differently, the same term always in the same way, when possible in a way that respects connotations from non-technical language. The sexagesimal place value numbers are rendered according to a system where ´ indicates decreasing, ` increasing order of sexagesimal magnitude and ° "order zero". These indications have no counterpart in the written text, but the virtual absence of errors shows that the authors of the texts must have been aware of them when they are needed (in homogeneous problems they are not).

[30] Such figures are never present on the tablets – the only diagrams which occasionally turn up are such as illustrate the statement (but even these are rare). It would indeed be very inconvenient to trace lines which then had to be deleted on the tablet; a likely support is sand strew on a paved floor, for instance the courtyard of the school.

The projection (with appurtenant rectangle) is broken into "moieties" – "natural halves"[31] – and the outer half moved around so that the two moieties together with the original square form a gnomon. Together, the two moieties "hold" a rectangle (in the actual case, a square) of area 30´×30´ = 15´ ($^1/_2 \times ^1/_2$ = $^1/_4$). This is "appended" (an asymmetric, concrete joining operation – the other "addition") to the gnomon, giving an area 1 for the completed square. "Close by" this square area 1, 1 "is equal" (namely, as side of the square). Removing the moiety 30´ that was moved around, we are left with 30´, which represents the side of the original square (the "confrontation").

As we see, no explicit arguments are offered for the correctness the procedure; but as we also notice, this correctness seems obvious – the approach is "naïve", as opposed to "critical", critique being understood as investigation of the conditions and limits of the validity of the argument.[32] This is opposed to our ideology of how "rigorous" mathematics should be made – but not too different from much of what is actually done: think of Georg Cantor's theory of sets, today justly identified as "naïve". Closer to the level of our text, we may look at how we solve the corresponding problem in numerical algebra (excluding negative numbers, which the Babylonians did not possess):

$$
\begin{aligned}
x^2 + 1\cdot x = {}^3/_4 \quad &\Leftrightarrow \quad x^2 + 1\cdot x + ({}^1/_2)^2 = {}^3/_4 + ({}^1/_2)^2 \\
&\Leftrightarrow \quad x^2 + 1\cdot x + ({}^1/_2)^2 = 1 \\
&\Leftrightarrow \quad (x + {}^1/_2)^2 = 1 \\
&\Leftrightarrow \quad x + {}^1/_2 = \sqrt{1} = 1 \\
&\Leftrightarrow \quad x = 1 - {}^1/_2 = {}^1/_2 \ .
\end{aligned}
$$

We *could* argue for the validity of each step on the basis of axioms (Euclid's, or a more recent version): "if equals be added to equals, the wholes are equal", etc. However, we rarely do it, we too are satisfied by "seeing" that everything is correct.

The same equation transformations illustrate why the interpretation of the texts as numerical algebra seemed convincing: the numbers we find in it coincide

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[31] A "natural half" is a half whose role could not possibly be taken over by any other fraction. It does not belong to the same family as, say, $^1/_3$, $^3/_5$ etc. The radius on a circle is the natural half of the diameter; that half of the base of a triangle that is multiplied by the height in order to yield the area is also natural.

[32] "Mann kann das Vermögen der Erkenntnis aus Prinzipien a priori die reine Vernunft und die Untersuchung der *Möglichkeit und Grenzen* derselben überhaupt die Kritik der reinen Vernunft nennen", as Kant opens the "Vorrede" to *Kritik der Urteilskraft* [*1956:* V, 237] (emphasis added).

precisely with those appearing in the Babylonian text. This is not always the case, sometimes the geometric foundation calls for deviations from the order of operations that seems natural in numerical/symbolic perspective; but often the agreement is perfect. In the present case, the difference between the two representations is most obvious in the appearance of phrases and operations that seem superfluous in an numerical perspective (the "projection", "in the inside"), and in the distinction between two different additive operations.

Beyond the shared naïve approach and agreement in the order of numbers, one further feature of the Babylonian procedure corresponds to what we do by equations: both are *analytic*, that is, they presuppose the existence of the solution and deal with it as with any normal segment respectively number in operations that eventually allows it to be disentangled from the relations within which it appears.[33]

The problem just discussed belongs on a tablet with 24 problems about one or several squares, all the others being more complex than the present one. First follows a problem where the side has been "torn out" from the area. "Tearing out" is a concrete subtractive operation, a removal of a magnitude from a larger magnitude of which it is a part – the inverse operation of "appending". The other subtractive operation, equally concrete, is comparison – the observation that one magnitude "goes so and so much beyond" another one. There is no subtractive counterpart of "accumulating" (the inverse of which is a splitting into constituents, an operation which turns up in a few texts). In order to tear out a segment from an area it therefore has to be imagined as a "broad line", a line provided with a virtual breadth – a notion of which there are also other traces in the corpus.[34]

In symbolic writing, problem #2 of the tablet is thus:

$$\square(s)-s = 14`30\ ,$$

which is interpreted geometrically as

$$\sqsubset\!\sqsupset(s,s-1) = 14`30$$

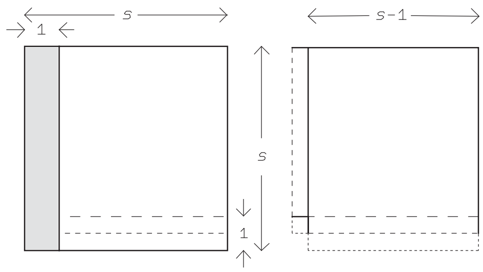
Once more, we are thus confronted with a rectangle, of which we know the area and the difference between the sides. The solution follows the same pattern, only this time we need to put back the piece we moved around in order to find *s*.

[33] Actually, analysis defined in this way is always naïve, working with entities whose existence has not been and cannot be argued for unless by means of a final synthesis. *Some* Babylonian texts contain a final proof (in the sense of check), which may be said to represent this synthesis.

[34] The notion of "broad lines" and its presence in a number of mathematical cultures is discussed in [Høyrup 1995].

Problem #3 of the text introduces a new challenge, namely coefficients. The statement runs as follows:[35]

> **9.** The third of the surface I have torn out. The third of the confrontation to the inside
> **10.** of the surface I have appended: 20´ is it.



The procedure of BM 13901 #2.

In approximate symbolic translation:

$$\square(s)-{}^{1}\!/_{3}\,\square(s)+{}^{1}\!/_{3}\,s = 20' \;,$$

or, simplified,

$${}^{2}\!/_{3}\,\square(s)+{}^{1}\!/_{3}\,s = 20' \;.$$

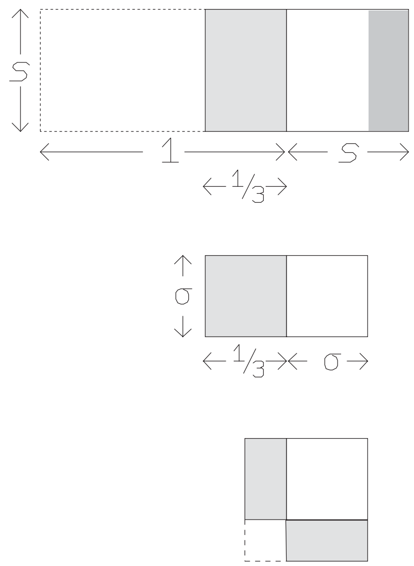
The non-unitary coefficient of *s* is no serious difficulty, we just need to replace the projection 1 by one third of it. The real challenge consists in the non-normalized character of the problem. In order to get around this difficulty, a change of scale in one direction is introduced, corresponding to the transformation

$$\square({}^{2}\!/_{3}s)+{}^{1}\!/_{3}\cdot({}^{2}\!/_{3}s) = {}^{2}\!/_{3}\cdot 20'$$

or, introducing $\sigma = {}^{2}\!/_{3}s$

$$\square(\sigma)+{}^{1}\!/_{3}\cdot\sigma = 13'20'' \;.$$

This gives us a normalized problem, and the text can go on as in #1. In the end, multiplication of σ by the reciprocal of ${}^{2}\!/_{3}$ gives *s*.



The procedure of BM 13901 #3.

#2 and #3 of the text are presented in the same naïve way as #1. However, two texts exist which give explicit didactical explanations. They are both from Susa, a peripheral area, where a need may have been felt to write down explanations which in the core area were only given orally (however, once the character of the Susa explanations are known, traces of similar expositions can be found also in texts from the core – see [Høyrup 2002a: 85].)[36]. They are

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35 Following [Høyrup 2002a: 53].

36 The mathematical Susa texts were published in [1961] par E. M. Bruins and M. Rutten. Unfortunately, the edition is problematic – not only the commentary but also the translation; often, even the transcriptions of Sumerograms into Akkadian are mistaken. In particular, the editors have completely overlooked the character of the didactical explanations. Here I follow the translation and interpretation given in [Høyrup 2002a: 85–95].

assumed to have been written toward the end of the Old Babylonian period.

Let us first look at TMS IX, an explanation of the basic tricks involved in the operation on second-degree problems.

#1

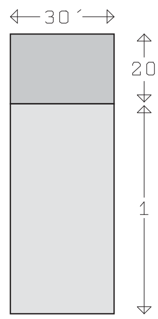
1.	The surface and 1 length accumulated, 4[0´. ¿30, the length,? 20´ the width.]
2.	As 1 length to 10´ ⸢the surface, has been appended,]
3.	or 1 (as) base to 20´, [the width, has been appended,]
4.	or 1°20´ [¿is posited?] to the width which 40´ together ⸢with the length ¿holds?]
5.	or 1°20´ toge⟨ther⟩ with 30´ the length hol[ds], 40´ (is) [its] name.
6.	Since so, to 20´ the width, which is said to you,
7.	1 is appended: 1°20´ you see. Out from here
8.	you ask. 40´ the surface, 1°20´ the width, the length what?
9.	[30´ the length. T]hus the procedure.

#2

10.	[Surface, length, and width accu]mulated, 1. By the Akkadian (method).
11.	[1 to the length append.] 1 to the width append. Since 1 to the length is appended,
12.	[1 to the width is app]ended, 1 and 1 make hold, 1 you see.
13.	[1 to the accumulation of length,] width and surface append, 2 you see.
14.	[To 20´ the width, 1 appe]nd, 1°20´. To 30´ the length, 1 append, 1°30´.[37]
15.	[¿Since? a surf]ace, that of 1°20´ the width, that of 1°30´ the length,
16.	[¿the length together with? the wi]dth, are made hold, what is its name?
17.	2 the surface.
18.	Thus the Akkadian (method).

#3

19.	Surface, length, and width accumulated, 1 the surface. 3 lengths, 4 widths accumulated,
20.	its [17]th to the width appended, 30´.
...	...



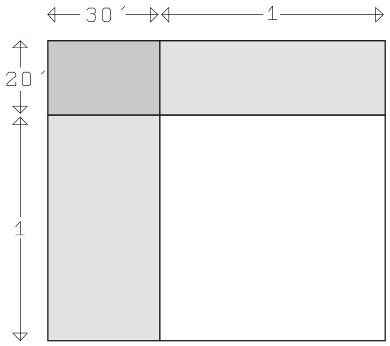
The configuration described in TMS IX #1.

Section #1 explains what to do when a segment and an area are accumulated – in the actual case, the length of a rectangular and its area; the dimensions are already known (30´ and 20´, respectively), without which (and in the absence of letter symbols or other naming abstract possibilities) it would be very difficult to formulate the explanation. This, as told in line 3, is

[37] The restitutions of lines 14–16 are somewhat tentative, even though the mathematical substance is fairly well established by the parallel in lines 28–31.

equivalent to appending a "base" 1 to the width 20´[38] – see the adjacent diagram. This gives a rectangle with sides 1°20´and 30´ (line 4) and hence area 40´, the same as the original sum. Lines 6–8 recapitulate, and in the end a kind of reverse calculation is made for control.

Section #2 is based on the same rectangle, still with known dimensions, but this time the area and *both* sides are accumulated, the sum being 1. Applying the same trick – applying a width 1 to each of the segments[39] – still gives us a geometrically meaningful configuration, but a rather unhandy one: a rectangle from which a square area is missing in a corner, see the figure; its area is still 1. But then the two extensions are "made hold", producing a complementary square of area 1×1 = 1, which fits exactly. This quadratic completion, spoken of twice as the "Akkadian method", results in a rectangle of area 1+1 = 2 and sides 1°30´ and 1°20´.



The configuration of TMS IX #2.

As we see, the explanations in both #1 and #2 are still fairly naïve, even though controls show that the result is indeed as it should be. The main purpose is to build up concepts and understanding, not to provide explicit demonstration from first principles.

After the two explanations follows a genuine problem, of which only the statement is quoted above. In symbolic translation, we are told that

$$\sqsubset\!\sqsupset(\ell,w)+\ell+w = 1\ ,\quad {}^{1}\!/\!_{17}(3\ell+4w)+w = 30´\ .$$

The first equation is precisely the one explained in #2. The second is of a kind which is explained in a different text, namely TMS XVI, though on somewhat simpler examples. Even this text contains several parts (2 indeed), the first of which runs like this:

1. [The 4th of the width, from] the length and the width to tear out, 45´. You, 45´
2. [to 4 raise, 3 you] see. 3, what is that? 4 and 1 posit,
3. [50´ and] 5´, to tear out, ⌈posit⌉. 5´ to 4 raise, 1 width. 20´ to 4 raise,
4. 1°20´ you ⟨see⟩, 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width,

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38 The translation "base" is tentative; it corresponds to what is suggested by the Sumerian sign combination ("something standing stably / permanently on the ground"), but it is found nowhere else.

39 This time without a name, perhaps because only one of them can be vertical and thus be understood as a "base".

to tear out,

5. from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, accumulate, 3 you see.

6. Igi 4 de[ta]ch, 15´ you see. 15´ to 2, lengths, raise, [3]0´ you ⟨see⟩, 30´ the length.

7. 15´ to 1 raise, [1]5´ the contribution of the width. 30´ and 15´ hold.

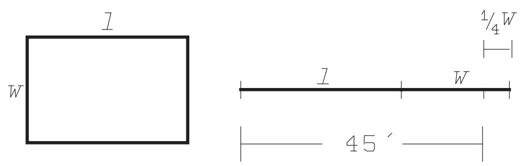
8. Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.

9. Igi 4 de⟨tach⟩, 15´ you see, 15´ to 3 raise, 45´ you ⟨see⟩, 45´ as much as (there is) of [widths].

10. 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1´ raise, 20´ you see.

11. 20´ to 45´ raise, 15´ you see. 15´ from $^{30}{}_{15}$´ [tear out],

12. 30´ you see, 30´ the length.



The situation of TMS XVI #1.

Once again, we are dealing with the 30´×20´-rectangle, but this time only with the sides. In symbolic translation, we have the equation

$$(\ell+w)-{}^{1}\!/_{4}\,w = 45´ \; .$$

As a first step, we are told to multiply the right-hand side 45´ by 4. The outcome is 3, and the text asks for the meaning of this number. In order to answer, it multiplies each member to the left, finding that $4\ell+(4-1)w = 4\ell+w$ is also 3 (lines 2–8). Then it goes backwards, finding the reciprocal of 4 ("detaching" its igi), which is 15´, and multiplying[40] the coefficients 3 and 4 of the new equation; it thus finds that those of the original one – "as much as (there is) of" lengths respectively widths – are 45´ and 1 (actually, the last number is not calculated, it is probably too obvious that 4 multiplied by its reciprocal is 1). Multiplying the length and the width by their respective coefficients gives the contribution of each to the equation, namely 20´ and 15´. Removing the latter from the sum 45´ (written in a non-standard way corresponding to what was memorized ["held"] in line 7) is seen to give the

[40] The operation is "to raise", a multiplication used when some consideration of proportionality is involved. The origin of the term is in volume calculation, where the basis of a prismatic body is "raised" from the default thickness of 1 cubit to the real height.

former, as it should.[41]

Even here, as we see, there is no deductive proof, but instead a systematic introduction of concepts and training of intuitive understanding.

If we return to TMS IX #3, its second equation

$$^{1}/_{17}(3\ell+4w)+w = 30'$$

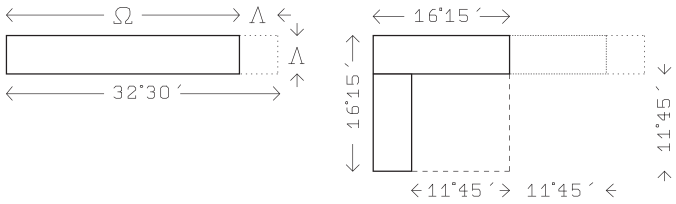
is multiplied by 17, just as taught in TMS XVI, which produces an equation

$$3\ell+21w = 8°30' \,.$$

Now both equations are transformed so as to deal with the length λ and the width ω "of 2 the surface" ($\lambda = \ell+1$, $\omega = w+1$):

$$3\lambda+21\omega = (3+21+8°30') = 32°30' \,, \quad \sqsubset\!\sqsupset(\lambda,\omega) = 2 \,.$$

Next the same stratagem is used as in BM 13901 #3 (but in two dimensions), and it is found that $\sqsubset\!\sqsupset(3\lambda,21\omega) = 3{\cdot}21{\cdot}2 = 2`6$ – that is, we have to find the sides of a rectangle $\sqsubset\!\sqsupset(3\lambda,21\omega)$ from the area (2`6) and the sum of the sides (32°30′). This is a standard problem, which is solved by a cut-and-paste procedure similar to but different from the one used when the area and the difference between the sides is known – cf. the figure ($\Lambda = 3\lambda$, $\Omega = 21\omega$). Finally, of course, first λ and ω and then ℓ and w are determined.



The transformed system of TMS IX #3.

It should be obvious that a complex calculation like this could never have been constructed "by trial and error" or "empirically" as sometimes maintained by those who know only BM 13901 #1 or similar simple cases. Without insights like those trained in TMS IX #1–2 and TMS XVI #1 (and quite a few more, for which we have not had the luck to find the texts), they could neither have been constructed nor solved

[41] Line 10 presents a small enigma, if we remember that the indication ′ of order of magnitude is not found in the original. Why should a "true width" 20 be multiplied by 1 so as to give "the width"? The translation presupposes that the "true width" is 20 n i n d a n = 120 m (and the true length 30 n i n d a n = 180 m). This is adequate for a real field but unhandy in the school-yard, where instead 20′ and 30′ (2 m and 3 m) would fit perfectly. This could be the reason that the standard "school rectangle" is $\sqsubset\!\sqsupset(30',20')$.

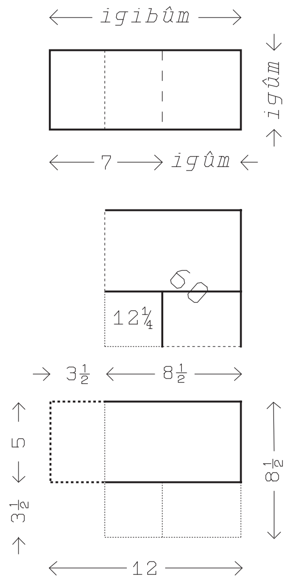
Our final text example will be YBC 6967:[42]

Obv.

1. [The *igib*]*ûm* over the *igûm*, 7 it goes beyond
2. [*igûm*] and *igibûm* what?
3. Yo[u], 7 which the *igibûm*
4. over the *igûm* goes beyond
5. to two break: 3°30´;
6. 3°30´ together with 3°30´
7. make hold: 12°15´.
8. To 12°15´ which comes up for you
9. [1` the surf]ace append: 1`12°15´.
10. [The equalside of 1`]12°15´ what? 8°30´.
11. [8°30´ and] 8°30´, its counterpart, lay down.

Rev.

1. 3°30´, the made-hold,
2. from one tear out,
3. to one append.
4. The first is 12, the second is 5.
5. 12 is the *igibûm*, 5 is the *igûm*.



The procedure of YBC 6967.

Igûm and *igibûm* are Akkadian loanwords coming from Sumerian, meaning respectively "the reciprocal" and "its reciprocal". The problem thus deals with a pair of numbers belonging together in the table of reciprocals; their product is taken to be 60, not 1, and their difference is 7. As we see, the problem is not geometrical at all but numerical. However, in line 9 the product is spoken of as a "surface", even though a term for the numerical product was at hand. The situation is thus the mirror of what we do in analytical geometry, where we *represent* geometrical entities by numbers and then work on them in a numerical algebra. Here, numbers are *represented* by measurable segments and areas.

The situation – a rectangle where the area and the difference between the sides are known – is the one we already know from BM 13901 #1 and #2, and also the procedure is the almost same. Of course, since we deal with a rectangle, both sides have to be found. Here something remarkable happens. For the Babylonians, as for us, addition "naturally" precedes subtraction (cf. the order of BM 13901 #1 and #2). Here, however, the piece which was moved is torn our

[42] First published in [Neugebauer & Sachs 1945: 129], here following the translation in [Høyrup 2002a: 55*f*].

first and appended afterwards. The reason is close at hand: the *same* piece is involved, and in order to appended it must in principle first be at hand, that is, torn out.

This is not evidence of "a primitive mind not yet ready for abstraction". Earlier texts, for instance those from Eshnunna, in similar situations use the phrase "to one append, from one cut off" (in this order),[43] and then give the double result. What we see is the outcome of *critique*. Apparently, somebody during the development of the discipline has discovered that the original formulation is metatheoretically untenable, and insisted on introducing a meaningful alternative; however, not all schools were affected by this choice, even the late texts from Sippar use the formulation we know from Eshnunna.

This is not the only instance of critique we find in the texts. In BM 13901 #1 and TMS IX #1–2, as we remember, areas and segments were accumulated. Obviously, it seems impossible to "append" segments to areas, since this is a concrete and thus a meaningful operation. However, even in this case some early texts differ, "appending" segments directly to areas, that is, presupposing that they are "broad lines" provided in themselves with a default width of one length unit.[44] If we also take into account that the explicit width carries different names,[45] it seems plausible that this elimination of the default breadth is another secondary development, caused by rejection of the ambiguous conflation of dimensions[46] – that is, precisely, by metatheoretical *critique*.

What precedes is no exhaustive description of Old Babylonian "algebra". Making use of the representation principle (letting segments represent areas or volumes), it was able to formulate and solve not only quadratic but also biquadratic problems; by means of factorization or the table "equalside one appended" (see note 10), it could solve certain irreducible equations of the third degree; though obstructed by inadequate terminology, it might interchange the

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43 To "cut off" functions as a synonym for "tearing-out". Since it has no Sumerographic equivalent, it is probably a borrowing from Akkadian lay surveyors.

44 One text from Eshnunna (undated, but probably also from the early 18th century BCE) [Goetze 1951] and AO 8862 [Neugebauer 1935: I, 108–113], apparently one of the earliest "algebraic" texts from the south.

45 Two more possibilities, employed in the text YBC 4714, are "second width" or "alternate width", which however represent a coefficient different from 1; see [Høyrup [2002a: 125*f*, 135].

46 Similar to what Plato does in *The Laws* 819D–820B, the passage where he speaks about the scandal that most Greek believe that lines, surfaces and volumes are commensurate or approximately commensurate.

roles of unknowns and coefficients; etc. But what was presented should demonstrate its fundamental character, which we may sum up as follows:

– Old Babylonian "algebra" was analytic, like applied equation algebra;
– its procedures were based on insight, not on blind "empirical" rules;
– but this insight was mostly naïve, only occasionally tainted by critique;
– the didactics aimed at concept formation and intuitive understanding, not at theoretical demonstration;
– theory was indeed never made explicit: whatever theoretical insights the authors of the texts possessed never made it into writing. Growing out of the supra-utilitarian level of scribal cunning, even advanced Old Babylonian mathematics always had as its aim *to find the right number*, exactly as was the aim of applied scribal calculation.

A concise comparison

Comparison with the ancient Greek and Hellenistic world and its mathematics should be made on two levels: social support, and mathematical substance.

Social support first. Obviously, practical mathematics was not absent from the Greco-Hellenistic world, and inasfar as numbers are concerned, those engaged in practical mathematics far exceeded those who produced theoretical mathematics (and even those who studied it seriously enough to know what deductive proof is).[47] However, those responsible for practical calculation never possessed the cultural hegemony which had characterized Mesopotamian calculator-scribes until the Old Babylonian period; even that vague reflection of the former glory of calculating mathematics which we find in Assurbanipal's boasting of being able to find reciprocals and to multiply is absent from classical culture. Greek and Hellenistic mathematics, to the extent it was accepted in elite culture, was either theoretical mathematics or somehow connected to post-Pythagorean, gnosticizing search for wisdom. The latter current certainly drew on the knowledge of practitioners – see [Høyrup 2001]. Yet it never admitted its debts: how, indeed, could "wisdom" ever have come from working people?

Theoretical mathematics may also have drawn some inspiration from practical traditions, but at large distance from the sources that have come down to us. That is simply to be expected, and agrees with what is told by Aristotle and Eudemos[48] – who however may have known even less than we do and may

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47 Cf. [Netz 2002] and [Asper 2009].

48 Aristotle, *Metaphysics* A, $980^{b}26$–$981^{a}12$ (quite unspecific, about practical knowledge preceding theoretical reflection on it) and Eudemos as used by Proclos, *Commentary on*

therefore also have reconstructed the process from what is to be expected.

However, exactly concerning the so-called “geometric algebra” of *Elements* II (not its use by Apollonios!) we may advance beyond expectations and “rational reconstructions”. Already in [1963], Neugebauer had given up the belief (if he ever held it) that the Greek learned directly from the Babylonians, and he now asserted that what we know from the Old Babylonian texts had become common knowledge in the whole Near East by the mid-first millennium, so there was no need for the Greeks to read cuneiform.

This, however, is still in need of further revision. Firstly, it goes by itself, the geometric reinterpretation of the Babylonian technique eliminates the proposed translation from arithmetic into geometry. What the Greek mathematicians may have encountered and have been inspired by was not a numerical algebra but a geometrical technique, much closer to what we find in *Elements* II. Certainly, this technique dealt with a geometry of measurable segments, which is of course different from what presents itself in surviving Greek theoretical geometry – but Aristotle is our witness that in his time, Greek theoretical geometry might still consider measurable entities.[49]

In any case, Greek geometers cannot have encountered the sophisticated Old Babylonian technique, with its use of freely chosen coefficients, factorization, representation of areas and volumes by segments, etc., since this technique had died with Old Babylonian scribal culture.

Closer analysis of what is found in *Elements* II (and in various other Greek theoretical works, from Euclid’s *Data* to Diophantos’s *Arithmetic*), reveals that nothing pointing back toward the Near Eastern tradition goes beyond the collection of surveyor’s riddles, which can indeed be shown to have been around

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the First Book of Euclid’s Elements 64.18–65.7, speaking (but still unspecifically) about the beginning of geometry in Egyptian mensuration and of arithmetic with Phoenician merchants

49 *Metaphysics* M 1078[a]25–29, trans. Ross in [Barnes 1984]:

> The same account may be given of harmonics and optics; for neither considers its objects qua light-ray or qua voice, but qua lines and numbers; but the latter are attributes proper to the former. And mechanics too proceeds in the same way. Thus if we suppose things separated from their attributes and make any inquiry concerning them as such, we shall not for this reason be in error, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the propositions.

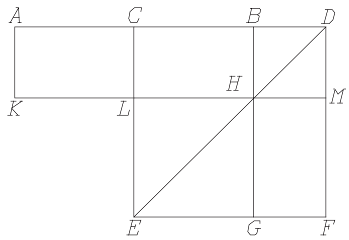
The vicinity of harmonics, optics and mechanics (as well as the whole topic of the passage) confirms that *theoretical geometry* is really in Aristotle’s thought.

in the first millennium CE and hence also in the mid-first millennium BCE.[50]

"Translation", moreover, involves not only translations of words and concepts but also a shift of aim. Riddles are problems and ask for a solution; nothing similar is found in *Elements* II. The Euclidean propositions may at most be characterized as the analogues of algebraic *identities*, and of such things we have no evidence in the supposed source tradition.

What is really at stake can be seen if we compare the cut-and-pastes procedure of BM 13901 #1 and YBC 6967 with *Elements* II.6. The proposition [trans. HEath 1926: I, 385] states that

> If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line.



The diagram of *Elements* II.6.

As we recognize, "the rectangle contained by the whole with the added straight line and the added straight line" corresponds to the rectangle for which we know the difference between the sides, while "the square on the half" is the quadratic complement.

However, Euclid does not proceed by cutting, moving and pasting. His proof starts by constructing the square *CEFD* and drawing the diagonal *DE*. Next through *B* the line *BHG* is drawn parallel to *CE* or *DF* (*H* being the point where the line cuts *DE*) and through *H* the line *KM* parallel to *AB* or *EF*. Finally, through *A* the line *AK* is drawn parallel to *CE* or *DF*.

Now the diagram is ready, and with reference to the way the construction was made ⊏⊐*AL* is shown to equal ⊏⊐*HF*. Adding ⊏⊐*CM* to both, the gnomon *CDFGHL* is seen to equal ⊏⊐*AM*. Further addition of □*LG* shows that ⊏⊐*AM* together with □*LG* equals □*CF*, as stated in the theorem.

So, all in all, Euclid (and his source, since this part of the *Elements* is probably borrowed wholesale from earlier theoretical geometry) is able to show, on the basis of definitions, postulates and common notions, that everything is as it should be, provided that angles are really right according to the definition, etc. That is, he presents us with a *critique* of traditional mensurational reason.

The whole sequence *Elements* II.1–10 can be shown in a similar manner to be critiques of the techniques used to solve the traditional riddles. As pointed

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[50] Details in [Høyrup 2001].

out by Ian Mueller [1981: 301], prop. 4–7 serve later in the *Elements* (not least in book X), whereas the others are never referred to again: their substance was apparently so familiar that it needed not be mentioned explicitly once its reliability had been confirmed. Obviously, the whole sequence establishes nothing new – but once the tools had been validated, they could serve further exploration, for instance in the *Conics*.

So, reformulated in various ways, Neugebauer's old idea to link Old Babylonian "algebra" with Greek "geometric algebra" can still be upheld.

The link, however, was between very different mathematical practices. We may return to the characteristics of Old Babylonian algebra as listed above:

– It was analytic, like applied equation algebra;
– its procedures were based on insight, not on blind "empirical" rules;
– but this insight was mostly naïve, only occasionally tainted by critique;
– the didactics aimed at concept formation and intuitive understanding, not at theoretical demonstration;
– theory was indeed never made explicit: whatever theoretical insights the authors of the texts possessed never made it into writing. Growing out of the supra-utilitarian level of scribal cunning, even advanced Old Babylonian mathematics always had as its aim *to find the right number*, exactly as was the aim of applied scribal calculation.

Beginning with the last point, Greek mathematical theory was not supra-utilitarian, it did not try to find the right number. Even Diophantos' *Arithmetic*, built up around number problems and neglecting the philosophical prohibition of fractions, mostly tries to find *one* possible solution to indeterminate problems, not *the right* solution.[51] *Problems* were certainly not absent from Greek geometry but perhaps a no less important activity than the production of theorems to the theoreticians – they were the medium through which one could demonstrate his skill (and the failing skill of competitors for glory). But problem solutions had to be *demonstrably* true.

So, not only when establishing theorems but also when dealing with

[51] To be true, one text from Susa deals with indeterminate problems of the first degree [Høyrup 1993]. In Diophantos. however, indeterminate problems are the rule, and determinate problems the rare exception – concentrated moreover in book I, which collects abstract variants of traditional "recreational" riddles and is therefore only determinate when the models are so.

problems, Greek theoretical mathematics was critical.[52] This does not mean, as we know, that the results of the Greeks would not be submitted to further critique in more recent centuries – critique is never definitive; but the Greek theoretical mathematicians dug until the point where *they* thought they had found firm ground (and when needed because critique ended up in circles, they established a postulate on which they could build[53]).

Since the Renaissance, it is a recurrent complaint that Euclid was not pedagogical but tried to hide the analysis which must certainly be behind his proofs. Clearly, the *Elements* do not aim at establishing an intuitive grasp of concepts. This may have many reasons, and one of them could easily be to scare away the insufficiently gifted – as Wilbur Knorr [1983; precise wording and page reference to be inserted] once asserted, *Elements* X is not meant for fun but for repression. But there may also be a metatheoretical reason: as pointed out in note 33, analysis is generally naïve, it makes use of entities whose existence has not yet been established. This is clearly illustrated by the twin sister of analysis: the indirect proof, which can indeed be characterized as analysis gone awry, analysis which ends up by showing that the entities it deals with cannot exist. The metatheoretical reason to avoid analysis may thus also explain (or be part of the explanation) that indirect proofs were not used nearly as often as we might find convenient.

So, Greek theoretical mathematics differed from Old Babylonian mathematics in many respects. But the differences are not accidental, they turn out to constitute a strongly connected network, defining Greek theoretical mathematics *as a practice* – just as growing out from supra-utilitarian mathematics determines Old Babylonian advanced mathematics as *a different practice*.

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[52] Within the "more physical" mathematical disciplines (the "mixed mathematics" of later times) the situation was not as clear-cut; I speak about theoretical arithmetic and, in particular, geometry.

[53] Cf. Aristotle's reference in *Analytica posteriora* I, 65[a]8–9 [trans. Barnes 1984] to "those persons [...] who suppose that they are constructing parallel lines [but] fail to see that they are assuming facts which it is impossible to demonstrate unless the parallels exist". Accepting that there was no way out of the circle, Euclid or a predecessor introduced the parallel postulate.

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